SJA MATHEMATICS CONTEST I

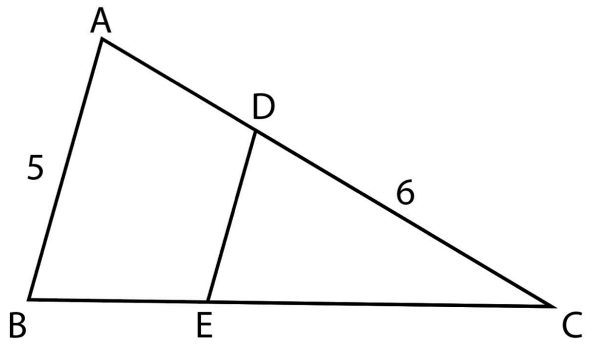
April 5, 2024

ADVANCED INDIVIDUAL ROUND

1. Let n be the smallest positive integer that must be multiplied to 2024 to make it a perfect square. Find the sum of the digits of n.

   (A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

2. Points D and E are on AC and BC, respectively, so that DE is parallel to AB, as shown. Triangle ABC has $AB = 5$, $DC = 6$, and $AD = DE$. What is the length of AD?



   (A) $2\sqrt{13}$ **(B)** $1 + \sqrt{39}$ **(C)** $2 + 2\sqrt{39}$ **(D)** $3\sqrt{39}$ **(E)** $-3 + \sqrt{39}$

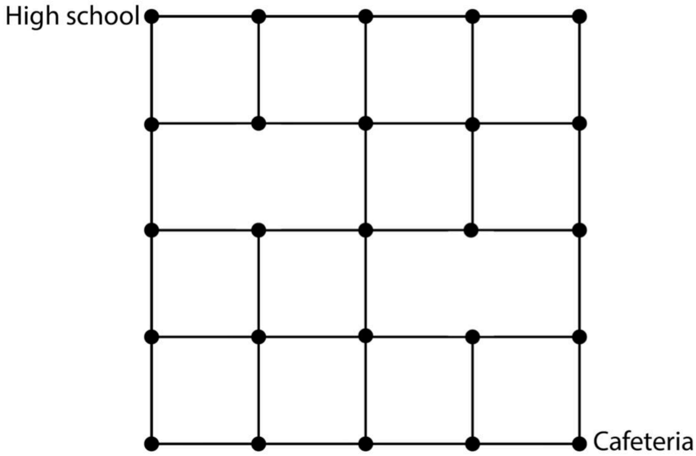
3. There are 2 blue books, 3 green books, and 4 red books. If the books with the same color are considered the same, in how many ways can Hilly stack those 9 books?

   (A) 1240 **(B)** 1250 **(C)** 1260 **(D)** 1270 **(E)** 1280

4. There are four consecutive integers a, b, c, and d. If $a^2 - d^2 = 84$, find the value of $\sqrt{b^2 + c^2}$. $(a > b > c > d)$

   (A) 5 **(B)** $\sqrt{34}$ **(C)** $6\sqrt{2}$ **(D)** $2\sqrt{21}$ **(E)** 10

5. Hilly wants to walk down from the high school to the cafeteria. If Hilly can only move down and to the right, how many different paths exist between the high school and the cafeteria?



(A) 34 **(B)** 40 **(C)** 48 **(D)** 68 **(E)** 70

6. An operation has a property where $x \diamondsuit y \triangle z = \frac{y}{xz} + z$ for all positive integers x, y, and z. The maximum value of $a \diamondsuit b \triangle c + b \diamondsuit c \triangle a + a \diamondsuit c \triangle b$, when reduced to its simplest form, is $\frac{\alpha}{\beta}$. Find $\alpha + \beta$.

(A) 112 **(B)** 159 **(C)** 217 **(D)** 281 **(E)** 325

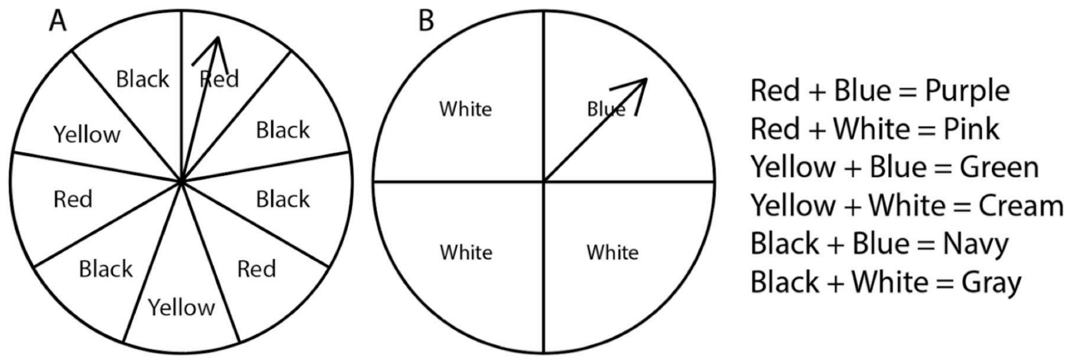
7. Solve for the smallest positive integer N that satisfies the equation $N + 7 \equiv 6 \pmod{10}$.

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

8. Let n be the base-8 representation of a positive integer k (k is in base-10 representation), where k is a perfect square. If k is less than 100, find the sum of all possible n.

(A) 300 **(B)** 303 **(C)** 309 **(D)** 393 **(E)** 501

9. Hilly has two spinners A and B. She spins each spinner once and mixes the two colors that she gets. For each spinner, the arrow is equally likely to stop on any one of the sectors sown. The probability of getting either pink, green, or navy, when reduced to its simplest form, is $\frac{m}{n}$. What is $m + n$? (Refer to the diagrams and color mixing below)



(A) 10 **(B)** 17 **(C)** 20 **(D)** 25 **(E)** 36

10. Points (0,12), (9,0), and (a,b) form an equilateral triangle. What is the area of the circle that passes through all three points?

(A) 65π **(B)** $\sqrt{75}\pi$ **(C)** 75π **(D)** 80π **(E)** 85π

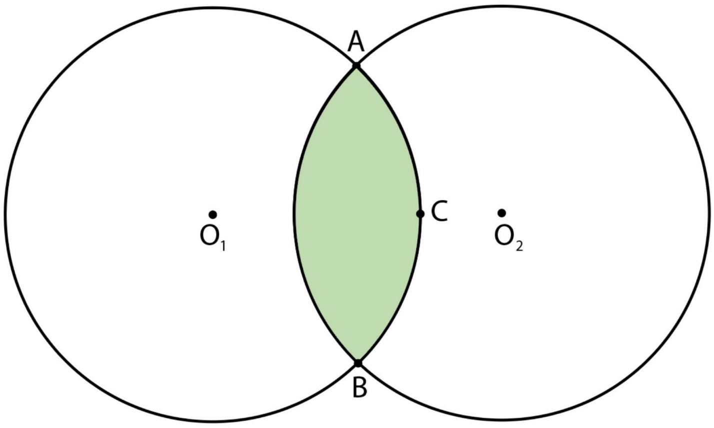
11. Find θ in the interval $0 < \theta < \pi$ that satisfies $4\sin(\theta) + \cos(2\theta) - 3 = 0$.

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{2}$ **(D)** $\frac{2\pi}{3}$ **(E)** $\frac{5\pi}{6}$

12. Martin and Kevin are playing a game. Martin chooses a real number from the interval [0,10] and Kevin chooses a real number from the interval [0,40]. Numbers are chosen independently and uniformly at random, and a player with a higher number wins the game. Assuming no ties, what is the probability of Martin winning the game?

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{16}$ **(E)** $\frac{1}{32}$

13. Two identical circles O_1 and O_2 are placed so that they overlap each other. Let A and B be the intersections of circles, and let C be the midpoint of arc AB, as shown. If $\frac{AC}{AO} = \sqrt{2 - \sqrt{2}}$ and $AO = 4$, what is the area of the shaded region?



(A) $6\pi - 8$ **(B)** $6\pi - 12$ **(C)** $8\pi - 16$ **(D)** $10\pi - 16$ **(E)** $12\pi - 12$

14. John, Lucy, and Elijah each have a six-sided dice in their hand. They roll the dice together repeatedly until everyone gets a prime number at the same time. However, if everyone gets a non-prime number at the same time, they stop rolling. What is the probability of them getting a prime number at the same time?

(A) $\frac{1}{8}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{5}{8}$ **(D)** $\frac{7}{8}$ **(E)** $\frac{1}{2}$

15. A simple decagon, which is a polygon with 10 sides, is inscribed in a circle with a circumference of 8π. Find the sum of the squares of all possible lengths by connecting two vertices of the decagon (Multiple equal lengths are counted as one possible length).

(A) 156 **(B)** 180 **(C)** $180\sqrt{2}$ **(D)** 192 **(E)** $192\sqrt{2}$