SJA MATHEMATICS CONTEST I

April 5, 2024

ADVANCED TEAM ROUND

1. If the greatest common divisor of $k \times 561$ and 2520 is 504, find the smallest positive integer value of k.

Answer: $\boxed{168}$

Explanation:

Prime factorization of $561 = 3 \times 11 \times 17$

Prime factorization of $504 = 2^3 \times 3^2 \times 7$

To have a common divisor of 504, $k \times 561$ must have $2^3 \times 3^2 \times 7$ as part of their factorization.

Since $561 = 3 \times 11 \times 17$, k must contain $2^3 \times 3 \times 7 = 168$. Because the question asks for the smallest positive integer value, $k = 168$.

2. Six runners from Pokpo and six runners from Shikmoolji race against each other. Then, four runners stumble during the race. The probability that the two are from Pokpo and the other two are from Shikmoolji, when reduced to its simplest form, is $\frac{m}{n}$. Find $m + n$ (The four runners were determined randomly)

Answer: $\boxed{16}$

Explanation:

The number of different combinations to choose two students among six Pokpo runners is ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$

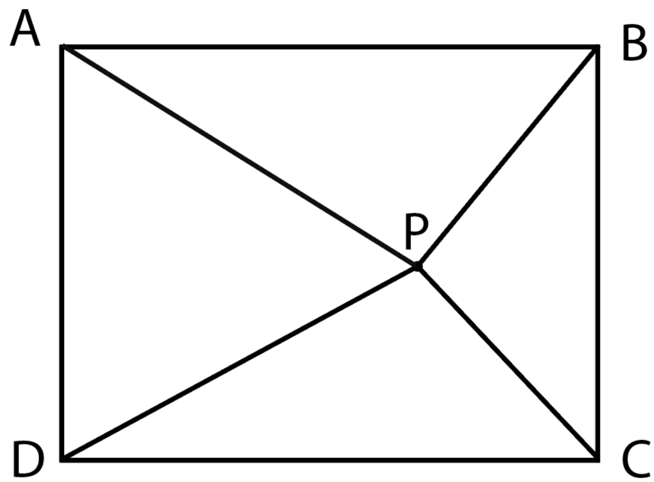
Likewise, the number of different combinations to choose two students among six Shikmoolji runners is ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$.

The number of different combinations of four students who will stumble during the race is ${}_{12}C_4 = \frac{12!}{(12-4)!\,4!} = 495$.

To get the probability where two are from Pokpo and two are from Shikmoolji, we have to multiply the combinations of choosing two Pokpo runners and two Shikmoolji runners. Then, we would divide the product by the combinations of choosing any four students who will stumble during the race.

$\frac{15 \times 15}{495} = \frac{5}{11}$ $m = 5$ and $n = 11$, so $m + n = 16$.

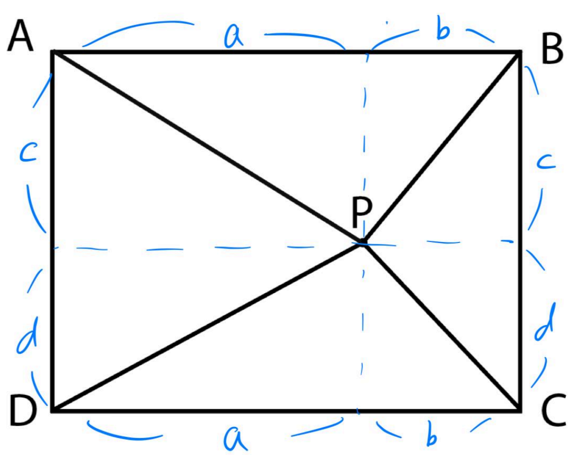
3. Point P is placed inside a rectangle ABCD where $AP = \sqrt{33}$ and $DP = 5$. If the lengths of BP and CP are integers, find the difference between BP and CP.



Answer: 2

Explanation:

Diagram for visualization:



As can be seen from the diagram, call the individual "sections" a, b, c and d. We get several equations from this layout:

$AP^2 = a^2 + c^2$, $BP^2 = b^2 + c^2$, $DP^2 = a^2 + d^2$, $CP^2 = b^2 + d^2$

We can mix these equations to say that:

$a^2 + b^2 + c^2 + d^2 = AP^2 + CP^2 = BP^2 + DP^2$, so we get the final needed equation: $AP^2 + CP^2 = BP^2 + DP^2$

Using this equation, $33 + CP^2 = 25 + BP^2$ which can be rewritten as $BP^2 - CP^2 = 8$. Now factor it: $(BP + CP)(BP - CP) = 8$. Because BP and CP are both integers, BP must be equal to 3 and CP must be equal to 1. Other combinations all require rational numbers.

Therefore, $BP - CP = 3 - 1 = 2$.

4. Hilly can either grab 4, 5, or 6 candies at once from a jar. Then, Hilly would put the candies inside her pocket. Assuming each step is distinct, how many ways can she fill her pocket with exactly 20 candies? (Hilly never puts the candy back into the jar)

Answer: 20

Explanation:

There are a total of 4 different combinations that distribute 20 candies:

[4, 4, 4, 4, 4], [5, 5, 5, 5], [4, 4, 6, 6], and [4, 5, 5, 6]. (Each number represents candies)

For [4, 4, 4, 4, 4], there is only one possible distribution method because all candies are of the same number.

For [5, 5, 5, 5], there is also one possible distribution method.

However, for [4, 4, 6, 6], the candies can also be distributed in [4, 6, 4, 6] or [6, 4, 6, 4] because the order that the candies are distributed matters. There are 4 "spots" to choose two 4s, so we use $_4C_2 = \frac{4!}{(4-2)!\,2!} = 6$. There are 6 different ways that [4, 4, 6, 6] is arranged. (You can also just count it, there are only 6)

For [4, 5, 5, 6], the process is similar, but now there are three distinct numbers. Because there are two 5s, the number of ways to put 5 in the arrangement is $_4C_2 = \frac{4!}{(4-2)!\,2!} = 6$. There are 4 and 6 left for each situation, so we need to multiply 2 to 6 and we get $2 \times 6 = 12$ different arrangements. (For example, Hilly can pull [4, 5, 5, 6] but also [6, 5, 5, 4])

Add all of the possibilities together, and final number of ways $= 1 + 1 + 6 + 12 = 20$.

5. When x^3 is given as $1 + 7i$ where $i = \sqrt{-1}$, solve for $x^6 + x^5 - 2x^3 - 2x^2 + \frac{50}{x} + 51$.

Answer: 1

Explanation:

$$x^3 = 1 + 7i$$

$$x^3 - 1 = 7i$$

$$(x^3 - 1)^2 = -49$$

$$x^6 - 2x^3 + 1 = -49$$

$$x^6 - 2x^3 + 50 = 0$$

$$x^5 - 2x^2 + \frac{50}{x} = 0 \qquad (5.1)$$

Substituting the value found from equation 5.1, $x^6 + x^5 - 2x^3 - 2x^2 + \frac{50}{x} + 51$ yields $x^6 - 2x^2 + 51$. To solve the expression, we need to find the value of $x^6 - 2x^2$.

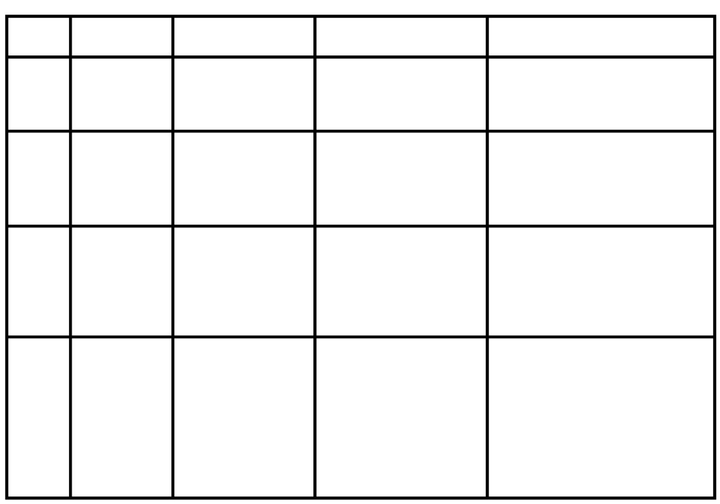
$$x^6 - 2x^2 = (x^3 - 1)^2 - 1$$

Since we already found that $(x^3 - 1)^2 = -49$,

$$x^6 - 2x^2 = -50$$

$x^6 - 2x^2 + 51 = (-50) + 51 = 1.$

6. How many rectangles exist in the grid?



Answer: 225

Explanation:

In order to make a rectangle, two horizontal lines and two vertical lines have to be chosen. Any two vertical lines and two horizontal lines will make a rectangle. Therefore, the total number of ways to choose any two vertical lines is ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$ because there are 6 vertical lines. Likewise, the total number of ways to choose any two horizontal lines is also ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$.

Simply multiply the possibilities together to form a rectangle, which means there are $15 \times 15 = 225$ different rectangles in the grid.

7. Hilly is packing her bag for a travel. She can pick any of the following: a water bottle, a notebook, a polaroid, headphones, a laptop, a watch, a sunscreen, and a radio. If Hilly packs at least two objects, in how many different ways can she pack her bag?

Answer: 247

Explanation:

There will be $2^8 = 256$ possible outcomes from Hilly's packing because she has two choices for each object (she can pick the object or not). However, we have to subtract the number of outcomes without any objects packed and outcomes with only one object packed because Hilly wants to pack at least two objects. Thus, there are $256 - 1 - 8 = 247$ different ways to pack her bag.

8. What is the maximum value of $a + b$ which makes both $\sqrt{125(a + 5)}$ and $\sqrt{18(b - 3)}$ positive integers? (a and b are positive integers less than 100).

Answer: 150

Explanation:

(1) Finding the maximum value of a

$\sqrt{125(a + 5)} = \sqrt{5^3(a + 5)}$. Prime factorization hints that $(a + 5)$ should be a product of 5 and a perfect square. The maximum value of $(a + 5)$ that satisfies the requirements is $5 \times 4^2 = 80$. (5×5^2 will not work because a should be less than 100). Let's verify our answer.

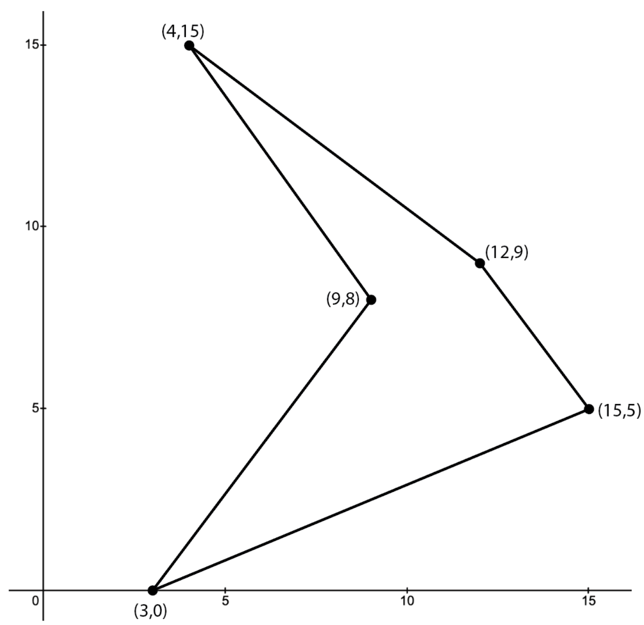
$\sqrt{125(80)} = \sqrt{5^3(5 \times 4^2)} = 5^2 \times 4$. This makes a positive integer. Since $a + 5 = 80$, $a = 75$.

(2) Finding the maximum value of b

$\sqrt{18(b - 3)} = \sqrt{3^2 \times 2\,(b - 3)}$. Prime factorization hints that $(b - 3)$ should be a product of 2 and a perfect square. The maximum value of $(b - 3)$ that satisfies the requirements is 72. Let's verify our answer: $\sqrt{3^2 \times 2 \times 72} = \sqrt{3^2 \times 2 \times 6^2 \times 2}$
$= 3 \times 2 \times 6$. This makes a positive integer. Since $b - 3 = 72$, $b = 75$.

$a + b = 75 + 75 = 150$.

9. Points (3,0), (9,8), (4,15), (12,9), and (15,5) form an irregular polygon. The sum of its area and perimeter can be written as $\frac{a}{2} + \sqrt{b}$ where a and b are positive integers. Find a + b.



Answer: 257

Explanation:

Although this problem can be approached by subtracting areas of triangles from the rectangle, there is much faster method called shoelace theorem to find the area.

Using the shoelace theorem,

$Area = \frac{1}{2}|3(8) + 9(15) + 4(9) + 12(5) + 15(0) - [0(9) + 4(8) + 12(15) + 15(9)$

$+ 3(5)]| = \frac{107}{2}$

Perimeter can be found using Pythagorean theorems:

$P = \sqrt{6^2 + 8^2} + \sqrt{5^2 + 7^2} + \sqrt{8^2 + 6^2} + \sqrt{3^2 + 4^2} + \sqrt{12^2 + 5^2} = 38 + \sqrt{74}.$

The Sum of the area and the perimeter = $\frac{107}{2} + 38 + \sqrt{74} = \frac{183}{2} + \sqrt{74}$

$a = 183$ and $b = 74$, so $a + b = 257$

10. $f(x)$ and $g(x)$ are given as $f(x) = -19x^2 - 9x - 1$ and $g(x) = -19x^2 - 9x + 2$. Find the value of $g(0) + g(1) + g(2) + ... + g(2024) - \{f(0) + f(1) + ... + f(2024)\}$.

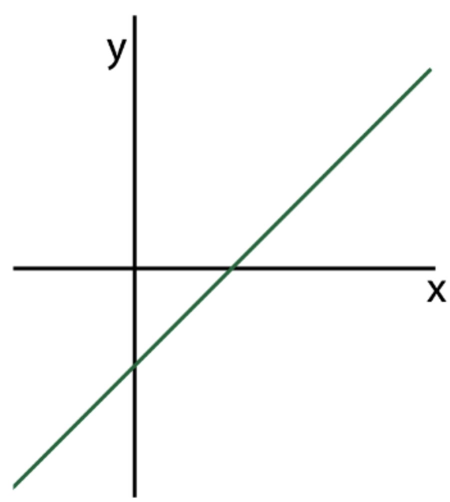
Answer: 6075

Explanation:

The expression can be rearranged to $[g(0) - f(0)] + [g(1) - f(1)] + ... + [g(2024) - f(2024)]$. Since $g(x) - f(x) = (-19x^2 - 9x + 2) - (-19x^2 - 9x - 1) = 3$,

$\sum_{n=0}^{2024} (g(n) - f(n)) = 3 \times 2025 = 6075.$

11. The linear function $-ax + by = c$ is shown on the right.



Which quadrant does $y = acx - ab$ not pass through?

Write your answer for ...

I Quadrant as 1
II Quadrant as 2
III Quadrant as 3
IV Quadrant as 4

Answer: 1

Explanation:

If we write the original equation $-ax + by = c$ into a slope-intercept form, it is: $y = \frac{a}{b}x + \frac{c}{b}$. By this, we can determine the signs of a, b, and c must easier.

There are two cases:

If a is negative, then b must also be negative because $\frac{a}{b}$ (the slope of the graph) is ultimately positive. This makes c positive, because the y-intercept, which is $\frac{c}{b}$, must be negative. In this case, the equation $y = acx - ab$ will have negative slope and negative y-intercept, so it will not pass Quadrant I.

If a is positive, then b must also be positive because $\frac{a}{b}$ (the slope of the graph) is ultimately positive. This makes c negative, because the y-intercept, which is $\frac{c}{b}$, must be negative. In this case, the equation $y = acx - ab$ will have negative slope and negative y-intercept, so it will not pass Quadrant I.

In any case, the equation $y = acx - ab$ will not pass Quadrant I, so the answer is 1.

12. N is a multiple of the sum of the divisors of 2024. If N has 64 divisors, find the smallest positive integer value of N.

Answer: 17280

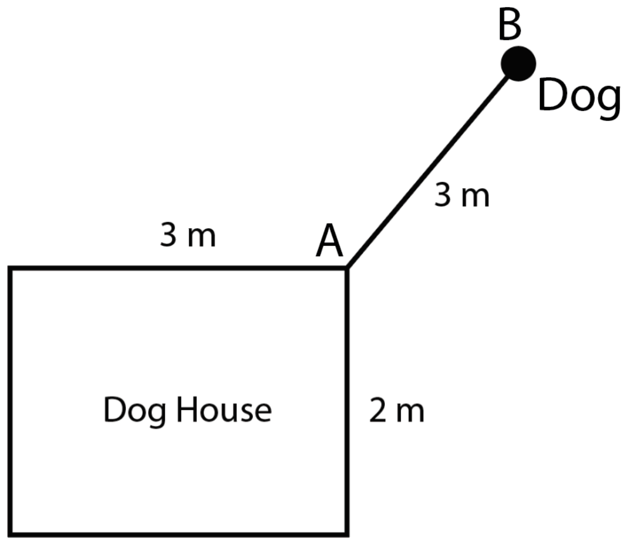
Explanation:

$2024 = 2^3 \times 11 \times 23$. So, the sum of the divisors of 2024 is $(2^0 + 2^1 + 2^2 + 2^3)(11^0 + 11)(23^0 + 23) = (15)(12)(24) = 4320$. This means that N is a multiple of 4320.

$4320 = 2^5 \times 3^3 \times 5$. Thus, 4320 has $(5 + 1)(3 + 1)(1 + 1) = 48$ divisors. Since the question asks for the smallest integer, let's multiply the smallest integer 2 as many as we need to find N. In order to have 64 divisors, we can multiply 2^2 to 4320.

$2^2(4320) = 2^2 \times 2^5 \times 3^3 \times 5 = 2^7 \times 3^3 \times 5$. There are $(7 + 1)(3 + 1)(1 + 1) = 64$ divisors. This shows that $2^2(4320)$ satisfies the condition. Thus, the smallest positive integer N is $4(4320) = 17280$.

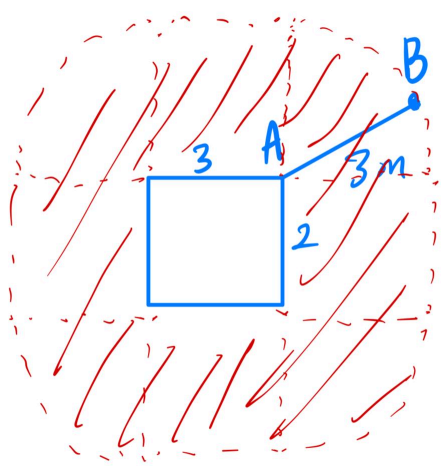
13. A dog's leash is 3m long and is connected to the outer walls of the 3m x 2m dog house, as shown. The dog can move anywhere as long as one end of the leash is connected to the outer walls of the dog house and another end is connected to the dog. The area outside of the dog house in which the dog can move around can be represented in the form $a\pi + b$, where a and b are positive integers. Find $a + b$.



Answer: 39

Explanation:

Diagram for visualization:



Simply need to find the red shaded as the leash is extended max. There are four Quarter circles with radius of 3, two rectangles with side length dimension 3×2, and two rectangles with side length dimension 3×3.

The four quarter circles can be combined into one, and the rectangle area can be easily found by multiplying. The total area is $3^2\pi + 2 \times 3 \times 3 + 2 \times 2 \times 3 = 9\pi + 30$.

$a = 9$ and $b = 30$, so $a + b = 9 + 30 = 39$.

14. When $2044 \times 2054 + 25 = n^2$, what is the sum of the digits of n? (n is a positive integer)

Answer: 15

Explanation:

Let $t = 2044$.

$$2044 \times 2054 + 25 = n^2$$

$$t(t + 10) + 25 = n^2$$

$$t^2 + 10t + 25 = n^2$$

$$(t + 5)^2 = n^2$$

$$(2049)^2 = n^2$$

$n = 2049$. The sum of the digits of n is $2 + 0 + 4 + 9 = 15$.

15. Complex number z is defined by $z = (x^2 - 20x + 91) + (x^2 - 10x + 21)i$, where $i = \sqrt{-1}$ and $\bar{z}$ is the complex conjugate of z. If $z \neq 0$, what is the value of x that satisfies $z = \bar{z}$?

Answer: 3

Explanation:

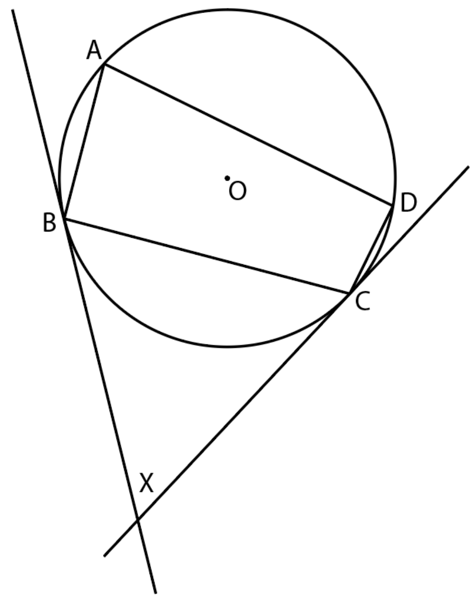
From factorization, the equation $z = (x^2 - 20x + 91) + (x^2 - 10x + 21)i$ can be rewritten as $z = (x - 7)(x - 13) + (x - 7)(x - 3)i$.

If $x = 7$, then the real part and imaginary part of the complex number z will both become 0. However, the question has stated that $z \neq 0$, so $x \neq 7$.

That leaves $x = 3$ because the imaginary part of z must be 0 in order for $z = \bar{z}$. If $x = 13$, that will make the imaginary part of z a nonzero number, thus making z not equal to $\bar{z}$.

Thus, the value of x that satisfies $z = \bar{z}$ is 3.

16. A quadrilateral $ABCD$ is inscribed in a circle O, as shown. $\angle DBC = 20^{\circ}$ and $\angle BCD = 110^{\circ}$. Two lines are tangent to the circle O at B and C, respectively. The tangent lines intersect to form $\angle X$. What is the sum, in degrees, of $\angle BAC$ and $\angle X$?



Answer: 130

Explanation:

$\angle BAD = 70^{\circ}$ because the sum of opposite angles of a quadrilateral inscribed in a circle is 180°.

$\angle DBC = \angle CAD = 20^{\circ}$ because they share the same arc. Thus, $\angle BAC = 70^{\circ} - 20^{\circ} = 50^{\circ}$.

$\angle BAC = 50^{\circ}$, so $\angle BOC = 2(50^{\circ}) = 100^{\circ}$. This is a part of the circle theorem which says if two angles share the same arc, the central angle is always twice of inscribed angle.

The external tangent lines form 90° when they meet with the circle. $\angle BOC + \angle X = 180^{\circ}$, so $\angle X = 180^{\circ} - 100^{\circ} = 80^{\circ}$.

The sum of $\angle BAC$ and $\angle X$ is $50^{\circ} + 80^{\circ} = 130^{\circ}$.

17. Ms. L distributes N number of candies to 3 students Tom, Jerry, and Teddy. She gives Tom 3 candies, Jerry 2 candies, and Teddy 1 candy. Then, she has 66 ways to distribute the candies that are left to three students. How many candies were there initially?

Answer: 16

Explanation:

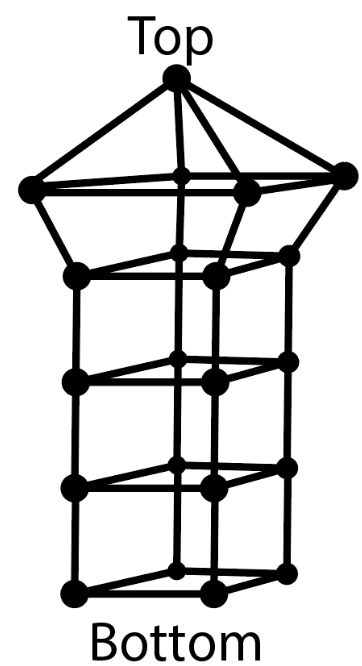
Let n be the number of candies left after Ms. L distributes candies to Tom, Jerry, and Teddy. So, $N = n + 6$. Since the candies are identical, we can use stars and bars to find the number of ways to distribute them. There are $C(n + 2, n)$ ways to distribute the candies that are left to three students.

$C(n + 2, n) = C(n + 2, n + 2 - n) = C(n + 2, 2) = \frac{(n+2)(n+1)}{2\times1}$

Since $\frac{(n+2)(n+1)}{2\times1} = 66$, $(n + 2)(n + 1) = 132$. The two consecutive integers that have a product of 132 are 11 and 12. This means that $n + 2 = 12$, so $n = 10$.

Thus, there were initially $N = 10 + 6 = 16$ candies.

18. The ant wants to walk down the tree house consisting of 21 vertices. If the ant only moves downward and horizontally, never revisiting the vertex, in how many different ways can the ant move down the tree house? (The ant starts from the top. When the ant reaches any four vertices at the bottom, the ant stops).



Answer: 9604

Explanation:

We will call each level 6th floor, 5th floor, 4th floor, and so on, starting from the top.

6th floor → 5th floor:

From the top, there are 4 ways to move down.

5th floor → 4th floor:

There are three options for the ant. It can go right, left, or down. The ant can move 3 times to the right, and 3 times to the left, and has 1 way to move down from where it's located. There are 7 ways to move down.

4th floor → 3rd floor:

Same as 5th floor → 4th floor.

3rd floor → 2nd floor:

Same as 5th floor → 4th floor.

2nd floor → 1st floor:

Same as 5th floor → 4th floor.

In total, there are $4(3 + 3 + 1)(7)(7)(7) = 9604$ ways for the ant to move down the tree house.

19. How many divisors do $(196 - 4 \times 14)^2 - 18(196 - 4 \times 14) - 63$ have?

Answer: 16

Explanation:

Let $196 - 4 \times 14 = a$.

Then, the equation given by the problem is $a^2 - 18a - 63 = (a - 21)(a + 3)$

$a = 196 - 56 = 140$. Put this value in the equation to get the number that we want. $(140 - 21)(140 + 3) = 119 \times 143 = 7 \times 17 \times 11 \times 13$.

This is the final number, but we need the number of divisors. The number of divisors is defined by $D(n) = (e_1 + 1)(e_2 + 1)$...., where e_1, e_2.. are the exponents of prime factors. For this problem, the factorization is $7 \times 11 \times 13 \times 17$, so the exponent of each prime factor is 1. So, the number of divisors is $(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 16$.

20. Hilly had 18 digit number. However, Hilly spilled water on the number, and the three digits were blurred.

If Hilly knows that the prime factorization of the number is $2^{11} \times 3^{8} \times 5^{4} \times 7^{6} \times 11^{2}$ and the leftmost blurred digit is greater than the sum of the other two blurred digits, find the sum of the three blurred digits.

Answer: 14

Explanation:

This question is all about number divisibility rules.

Call the leftmost hidden number a, the middle hidden number b, and the right hidden number c.

It is known by factorization that the 18 digit number must be divisible by 9. Denote that a number is divisible by 9 if the sum of the digits is divisible by 9. This means that $1 + 1 + a + 5 + 5 + 1 + 1 + b + 1 + 3 + 8 + c + 3 + 2 = 31 + a + b + c$ is divisible by 9 as well. This means there are two possibilities:

$a + b + c = 5$ or $a + b + c = 14$. $a + b + c$ cannot be 23 or higher because the maximum sum of a, b, c is 17.

Another number divisibility rule that we'll use is 11. A number is divisible by 11 if $|(Sum\ of\ the\ odd\ digits) - (Sum\ of\ the\ even\ digits)|$ is divisible by 11. Here the odd digits mean the numbers in odd number of digits, such as first digit, third digit, and so on. Same for the even digits.

Therefore, $|(1 + 5 + 1 + b + 3 + c + 2) - (1 + a + 5 + 1 + 1 + 8 + 3)|$, which is equal to $|(12 + b + c) - (19 + a)| = |b + c - a - 7|$, should be divisible by 11. The candidates for $b + c - a$ are 18, 7, -4, -15, and so on. However, a must be bigger than $b + c$ (and they are all positive one-digit numbers), so the only candidate possible is $b + c - a = -4$.

Now there are two possible sets of equations:

$\{a + b + c = 5$ and $b + c - a = -4\}$ or $\{a + b + c = 14$ and $b + c - a = -4\}$

Using a system of equations, the first set results in $2a = 9$, and this is not possible because a must be an integer.

The second set results in $2a = 18$, which means $a = 9$. This means that the second set satisfies the requirements.

By the second set, $a + b + c = 14$

Method 2: Multiply all the factors together to get the answer.