SJA MATHEMATICS CONTEST I

April 5, 2024

INTERMEDIATE TEAM ROUND

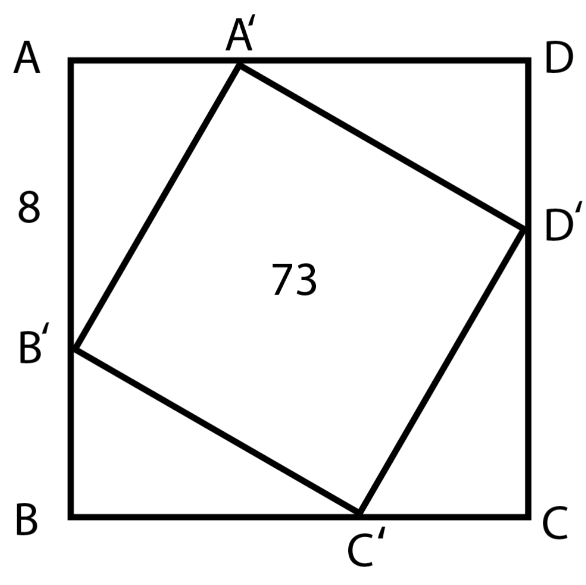
1. What is the least common multiple of 19 and 29?

Answer: 551

Explanation:

The LCM will be the smallest number that is the multiple of both 19 and 29. 19 and 29 are both prime numbers. Because the least common multiple of prime numbers is always the multiplication of those numbers themselves, the least common multiple of 19 and 29 is $19 \times 29 = 551$.

2. A smaller square $A'B'C'D'$ with an area of 73 is placed inside a bigger square $ABCD$ so that each vertex of the smaller square is touching one of the sides, as shown. If $AB' = 8$, what is the area of ABCD?



Answer: 121

Explanation:

$A'B'C'D'$ is a square, so it has equal lengths for every side. That gives us the side $A'B' = \sqrt{73}$. The triangle $\Delta AA'B'$ is a right triangle, so by the Pythagorean theorem the side $AA' = \sqrt{73 - 64} = \sqrt{9} = 3$.

Now, $\angle AB'A' + \angle BB'C' = 90$ and $\angle BC'B' + \angle BB'C' = 90$.

This means that $\angle AB'A' + \angle BB'C' = \angle BC'B' + \angle BB'C'$, and $\angle AB'A'$ is equal to $\angle BC'B'$. With the same logic, we can find that $\angle AA'B'$ it is equal to $\angle BB'C'$. With this information, $\Delta AA'B'$ is congruent to $\Delta BB'C'$ due to the ASA Triangle Congruence Theorem.

Because $\Delta AA'B'$ and $\Delta BB'C'$ are congruent to each other, $AA' = BB' = 3$. $AB = AB' + BB' = 8 + 3 = 11$, where AB is the side of the square $ABCD$. So, the area of square $ABCD$ is $11^2 = 121$.

3. $(b, -1)$ is the root of the following system of equations

$$\begin{cases} 0.01x + 0.2y = -0.09 \\ \frac{1}{3}x + \frac{2}{3}y = a \end{cases}$$

When $\frac{a}{b}$ is expressed as a decimal, what is the hundredth digit?

Answer: 7

Explanation:

First, we put $(b, -1)$ in the equations to find the value of b:

$$0.01b + 0.2(-1) = -0.09$$

Solve this equation, and the value of $b = 11$. Now we can find the value of a by substituting $(b, -1)$ into the second equation:

$$\frac{1}{3}(11) + \frac{2}{3}(-1) = a = 3$$

$\frac{a}{b} = \frac{3}{11}$, and if we divide by hand, we get $\frac{3}{11} = 0.2727...$ Therefore, the hundredth digit of $\frac{a}{b}$ is 7.

4. An improper fraction $\frac{B}{A}$ can be expressed as a mixed fraction $C\frac{1}{A}$. What is the value of $A \times B \times C$ when $B = 2$?

Answer: 2

Explanation:

$\frac{B}{A} = C\frac{1}{A}$, which means that $B = AC + 1$. This can be rewritten to $B - 1 = AC$, and using this $A \times B \times C = B(B - 1) = 2(2 - 1) = 2$.

5. There are 8 students in Mr. P's advisory. Mr. P randomly chooses 2 students to draw and 2 students to cut. If a student cannot perform more than one role, in how many ways can Mr. P assign roles to his students?

Answer: 420

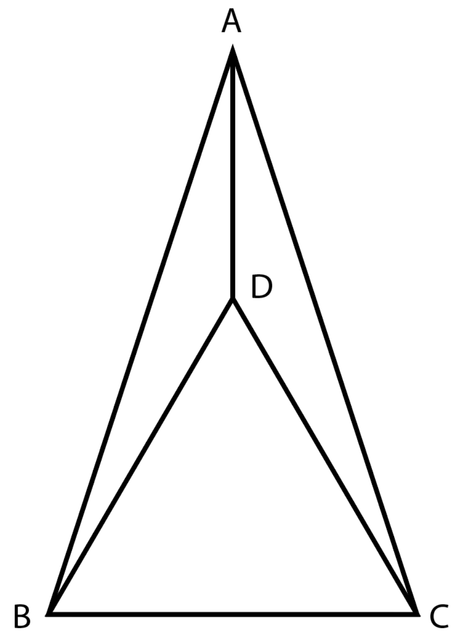
Explanation:

First, we need to determine the students who will draw. If we choose 2 random students to draw among 8 students, using the combinations formula, $C(8, 2) = \frac{8!}{2!(6!)} = 28 =$ (The number of different combinations we can choose 2 students to draw). So there are 28 ways to select the first 2 students to draw.

Then, we have to choose students who will cut. Since 2 students were already chosen to draw, we only have 6 students to randomly choose from. Very similar to selecting students who will draw, $C(6, 2) = \frac{6!}{2!(4!)} = 15 =$ (The number of different combinations we can choose 2 students to cut among 6 students left).

We have 28 ways to pick 2 students to draw and 15 ways to pick 2 students to cut. Multiplying the number of ways, we get $28 \times 15 = 420$ different ways for Mr. P to assign roles to his students.

6. In the diagram, triangle ABC is isosceles with AB = AC = 9 and triangle BCD is equilateral with a side length of 6. If $\sqrt{2} = 1.4$ and $\sqrt{3} = 1.7$, the area of triangle ABD is closest to which positive integer?



Answer: 5

Explanation:

Let us call the midpoint of BC as O. The length of $BO = \frac{6}{2} = 3$, and by the Pythagorean theorem the length of $DO = \sqrt{DB^2 - BO^2} = \sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$. With a similar use of the Pythagorean theorem, the length of $AO = \sqrt{AB^2 - BO^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$. Now that we have a length of DO and AO, the length of $AD = AO - DO = 6\sqrt{2} - 3\sqrt{3}$.

The area $\triangle ABD$ is simply $\frac{base \times height}{2}$, like any other triangle. If the base of $\triangle ABD$ is AD, the height will be BO. So $\frac{base \times height}{2} = \frac{BO \times AD}{2} = \frac{3(6\sqrt{2} - 3\sqrt{3})}{2} = \frac{3(6(1.4) - 3(1.7))}{2} = 4.95 \approx 5.$

(Note it is given from question that $\sqrt{2} = 1.4$ and $\sqrt{3} = 1.7$)

7.

97	125	84	95	95	90
98	90	105	81	85	x

The numbers above are the measured weights of 12 SJA students. If the mode and mean are the same, what is the value of x? (There is only one mode in the set)

Answer: 95

Explanation:

Mean of the dataset, or the average, is equal to the sum of all the numbers divided by the total number of data. Mean of the set $= \frac{97 + 125 + 84 + 95 + 95 + 90 + 98 + 90 + 105 + 81 + 85 + X}{12} = \frac{1045 + X}{12}$.

As the question stated, there should only be one mode in the set. Therefore, X can only be either 95 or 90. If X is any other number, then there will be two or three modes, so it is only possible for X to be 95 or 90.

If $X = 90$, mean $= \frac{1045 + 90}{12} = 94.58... \neq 90 =$ mode. Therefore, X cannot be 90.

On the other hand, if $X = 95$, mean $= \frac{1045 + 95}{12} = 95 =$ mode. It fits perfectly and thus $X = 95$ is the answer.

8. A kite with diagonals 8 and 4 has the same area as a square. What is the side length of that square?

Answer: 4

Explanation:

A kite is a quadrilateral which always has diagnols perpendicular to each other. Therefore this question's kite has the area of $\frac{8 \times 4}{2} = 16$. This means that the area of the square is 16. Since $(area\ of\ the\ square) = (side\ length\ of\ the\ square)^2$, the side length of the square is $\sqrt{16} = 4$.

9. How many ways are there to arrange letters H, I, L, and Y in the blanks below, so no same letters are placed next to each other?

___ ___ L ___ ___

Answer: 12

Explanation:

(Counting from left) The second empty spot and the third empty space can only be filled with H, I, and Y because L cannot go into either of those spots. That would make L placed next to each other. The order which letters are placed matters, so there are a total of ${}_3P_2 = \frac{3!}{(3-2)!} = 6$ different ways to place in second and third empty spots.

The first and fourth empty spots will have 2 letters of choice because 2 letters have already been used in the second and third spots. That means there are ${}_2P_2 = \frac{2!}{(2-2)!} = 2$ choices for those two spots.

To get the total number of ways, multiply the individual possibilities to get $6 \times 2 = 12$ different ways.

10. $5^7(2024)(20)(24)$ can be written as $N(10^8)$. What is the sum of the digits of N?

Answer: 21

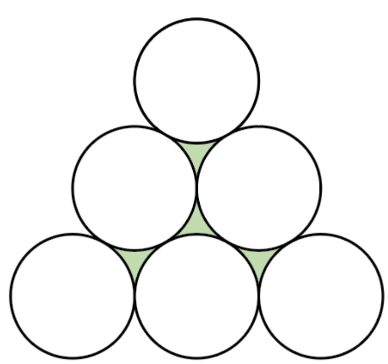
Explanation:

Prime factorization of $2024 = 2^3(11)(23)$. Prime factorization of $20 = 2^2(5)$. Prime factorization of $24 = 2^3(3)$. Using the factorizations, we can rewrite the original number as $5^7(2024)(20)(24) = 5^7(2^3)(11)(23)(2^2)(5)(2^3)(3) = (11)(23)(3)(2^8 \times 5^8) = 759(10^8)$.

The number $N = 759$.

The sum of digits of $N = 7 + 5 + 9 = 21$.

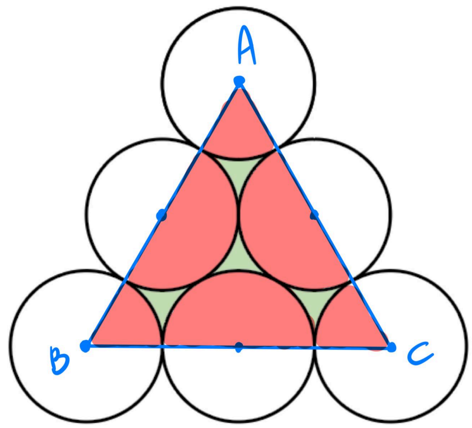
11. Six identical circles each having a radius of 4 are placed, so that the neighboring circles are tangent to each other. Connecting each center of the circles forms an equilateral triangle. The shaded area can be written as $a\sqrt{b} - c\pi$ where b is a positive prime integer. Find $a + b + c$.



Answer: 99

Explanation:

Diagram for visualization:



(Green shaded area) + (Red shaded area) = (area of triangle ABC)

The red-shaded area is simply two full circles when we add the angles up together (Equilateral triangles have an angle of 60 degrees). So, (red shaded area) $= 2 \times 4^2\pi = 32\pi$.

Using Pythagorean theorem, (Height of triangle ABC) $= \sqrt{16^2 - 8^2} = 8\sqrt{3}$.

(Area of ABC) $= \frac{height \times base}{2} = \frac{8\sqrt{3} \times 16}{2} = 64\sqrt{3}$.

Green shaded areas are what we are trying to find. The green shaded area will be the difference between the Area of ABC and the red shaded area. This means that the green shaded area is $64\sqrt{3} - 32\pi$. This makes $a = 64$, $b = 3$, and $c = 32$, so $a + b + c = 99$.

12.

$$|x - 1| + |x + 1| \leq 4$$

How many integers are possible for x?

Answer: 5

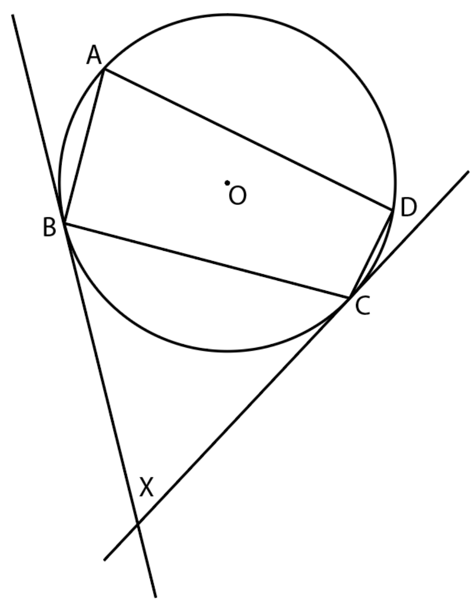
Explanation:

There can be three scenarios based on the domain of x:

First, $x \geq 1$. Then, the equation will become $(x - 1) + (x + 1) \leq 4$, since $(x - 1)$ and $(x + 1)$ are both positive numbers. Simplify it and you get $x \leq 2$. The integer values possible for x are 1 and 2 because we initially set the domain of x as greater than or equal to 1.
Second, $-1 < x < 1$. Then the equation becomes $-(x - 1) + (x + 1) \leq 4$ because $(x - 1)$ is negative but $(x + 1)$ is positive. Simplify it and you get $2 \leq 4$, which is always true no matter the value of x. We get the solution $x = 0$ because we said $-1 < x < 1$.
Lastly, $x \leq -1$. The equation becomes $-(x - 1) - (x + 1) \leq 4$, because both $(x - 1)$ and $(x + 1)$ are negative. Simplify it and you get $-2x \leq 4$, which gives solution of $x = -1, -2$.

Put all the solutions together and $x = -1, -2, 0, 1, 2$, a total of 5 different solutions.

13. A quadrilateral $ABCD$ is inscribed in a circle O, as shown. $\angle DBC = 20^{\circ}$ and $\angle BCD = 110^{\circ}$. Two lines are tangent to the circle O at B and C, respectively. The tangent lines intersect to form $\angle X$. What is the sum, in degrees, of $\angle BAC$ and $\angle X$?



Answer: 130

Explanation:

$\angle BAD = 70^{\circ}$ because the sum of opposite angles of a quadrilateral inscribed in a circle is 180°.

$\angle DBC = \angle CAD = 20^{\circ}$ because they share the same arc. Thus, $\angle BAC = 70^{\circ} - 20^{\circ} = 50^{\circ}$.

$\angle BAC = 50^{\circ}$, so $\angle BOC = 2(50^{\circ}) = 100^{\circ}$. This is a part of the circle theorem which says if two angles share the same arc, the central angle is always twice of inscribed angle.

The external tangent lines form 90° when they meet with the circle. $\angle BOC + \angle X = 180^{\circ}$, so $\angle X = 180^{\circ} - 100^{\circ} = 80^{\circ}$.

The sum of $\angle BAC$ and $\angle X$ is $50^{\circ} + 80^{\circ} = 130^{\circ}$.

14. Given $x + y = 2$ and $x^2 + y^2 = 6$, find the value of $x^3 + y^3$.

Answer: 14

Explanation:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$4 = 2xy + (x^2 + y^2)$$

$$4 = 2xy + (6)$$

$$xy = -1$$

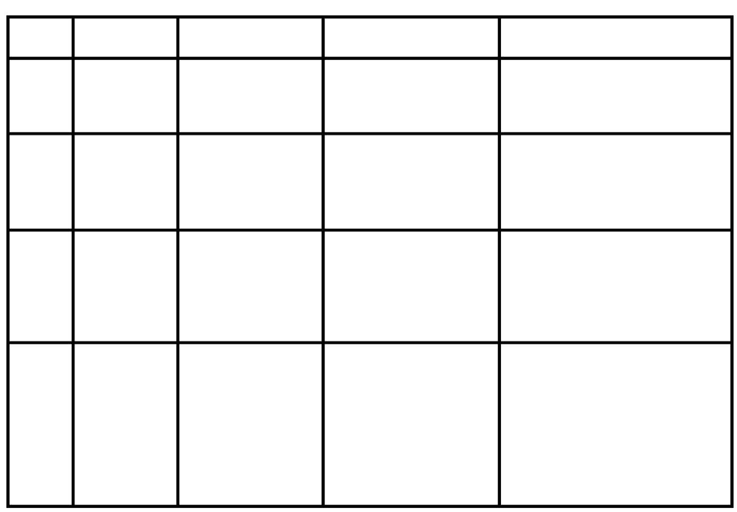
$$(x^2 + y^2)(x + y) = x^3 + x^2y + xy^2 + y^3$$

$$(6)(2) = x^3 + y^3 + xy(x + y)$$

$$12 = x^3 + y^3 + (-1)(2)$$

$$x^3 + y^3 = 14$$

15. How many rectangles exist in the grid?



Answer: 225

Explanation:

In order to make a rectangle, two horizontal lines and two vertical lines have to be chosen. Any two vertical lines and two horizontal lines will make a rectangle. Therefore, the total number of ways to choose any two vertical lines is ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$ because there are 6 vertical lines. Likewise, the total number of ways to choose any two horizontal lines is also ${}_6C_2 = \frac{6!}{(6-2)!\,2!} = 15$.

Simply multiply the possibilities together to form a rectangle, which means there are $15 \times 15 = 225$ different rectangles in the grid.

16. Hilly is solving for the following system of equations

$$\begin{cases} 2x + 3y = -1 \\ 4x + 5y = -1 \end{cases}$$

However, Hilly misread -1 in $2x + 3y = -1$ to another number and got $x = -4$. What was the number that Hilly read instead?

Answer: 1

Explanation:

Set the number that Hilly misread as a. We can consider that except for -1, Hilly correctly read the numbers.

If we plug $x = -4$ to the second equation, we get $4(-4) + 5y = -1$, therefore $y = 3$.

Insert the values $x = -4$ and $y = 3$ into the first equation (which Hilly misread) and you get $2(-4) + 3(3) = a = 1$, which is the number that Hilly misread.

17. There are two integers a and b so their greatest common divisor is 6 and their least common multiple is 60. What is the smallest possible value of $a + b$?

Answer: 42

Explanation:

Their greatest common divisor is $6 = 3 \times 2$, which means both a and b should have 3×2 as part of their prime factorization.

Their least common multiple is $60 = 2^2 \times 3 \times 5$, which means there should be additional 2 and 5 among a and b.

For example, a can be $2 \times 3 \times 5$ and b can be $2^2 \times 3$, which is actually the set that is the smallest possible value of $a + b = 2 \times 3 \times 5 + 2^2 \times 3 = 30 + 12 = 42$.

a can also be $2^2 \times 3 \times 5$ while b be 2×3. Although this combination also meets conditions, the sum will be $a + b = 2^2 \times 3 \times 5 + 2 \times 3 = 60 + 6 = 66$, which is much larger than the previous one.

It is possible to check for other combinations, but 42 will be the smallest sum.

18. Hilly the goalkeeper has a 40% chance of blocking the penalty kick. The probability of Hilly blocking 3 goals out of 5, when reduced to its simplest form, is $\frac{m}{n}$. Find $m + n$.

Answer: 769

Explanation:

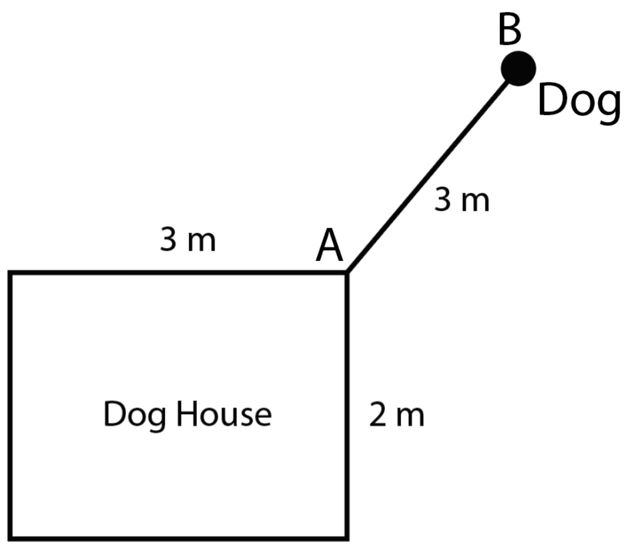
The probability that Hilly blocks the first 3 goals and fails to block the last 2 goals is $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{2^3 \times 3^2}{5^5}$.

However, there are multiple ways in which Hilly blocks 3 goals and fails to block 2 goals. For example, Hilly can block the first, third, and last goals, but fails to block the second and fourth goals. To determine the total number of different combinations that Hilly blocks, we need to know how many ways can Hilly block 3 goals among 5. Therefore we use combinations, and there are ${}_5C_3 = \frac{5!}{(5-3)!\,3!} = 10$ different ways.

The final probability that Hilly blocks any three goals and doesn't block the remaining 2 goals can be found by multiplying individual probabilities: $\frac{2^3 \times 3^2}{5^5} \times 10 = \frac{2^4 \times 3^2}{5^4} = \frac{144}{625}$.

$m = 144$ and $n = 625$, so $m + n = 144 + 625 = 769$.

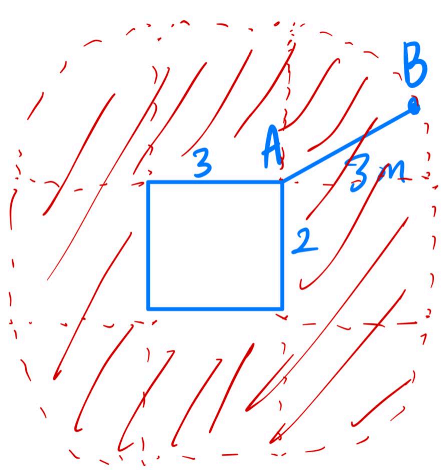
19. A dog's leash is 3m long and is connected to the outer walls of the 3m x 2m dog house, as shown. The dog can move anywhere as long as one end of the leash is connected to the outer walls of the dog house and another end is connected to the dog. The area outside of the dog house in which the dog can move around can be represented in the form $a\pi + b$, where a and b are positive integers. Find $a + b$.



Answer: 39

Explanation:

Diagram for visualization:



Simply need to find the red shaded as the leash is extended max. There are four quarter circles with radius of 3, two rectangles with side length dimension 3×2, and two rectangles with side length dimension 3×3.

The four quarter circles can be combined into one, and the rectangle area can be easily found by multiplying. Total area = $3^2\pi + 2 \times 3 \times 3 + 2 \times 2 \times 3 = 9\pi + 30$.
$a = 9$ and $b = 30$, so $a + b = 9 + 30 = 39$.

20. Hilly can either grab 4, 5, or 6 candies at once from a jar. Then, Hilly would put the candies inside her pocket. Assuming each step is distinct, how many ways can she fill her pocket with exactly 20 candies? (Hilly never puts the candy back into the jar)

Answer: 20

Explanation:

There are a total of 4 different combinations that distribute 20 candies:

[4, 4, 4, 4, 4], [5, 5, 5, 5], [4, 4, 6, 6], and [4, 5, 5, 6]. (Each number represents candies)

For [4, 4, 4, 4, 4], there is only one possible distribution method because all candies are of the same number.

For [5, 5, 5, 5], there is also one possible distribution method.

However, for [4, 4, 6, 6], the candies can also be distributed in [4, 6, 4, 6] or [6, 4, 6, 4] because the order that the candies are distributed matters. There are 4 "spots" to choose two 4s, so we use ${}_4C_2 = \frac{4!}{(4-2)!\,2!} = 6$. There are 6 different ways that [4, 4, 6, 6] is arranged. (You can also just count it, there are only 6)

For [4, 5, 5, 6], the process is similar, but now there are three distinct numbers. Because there are two 5s, the number of ways to put 5 in the arrangement is ${}_4C_2 = \frac{4!}{(4-2)!\,2!} = 6$. There are 4 and 6 left for each situation, so we need to multiply 2 to 6 and we get $2 \times 6 = 12$ different arrangements. (For example, Hilly can pull [4, 5, 5, 6] but also [6, 5, 5, 4])

Add all of the possibilities together, and final number of ways = $1 + 1 + 6 + 12 = 20$.