SJA MATHEMATICS CONTEST II

April 11, 2025

ADVANCED TEAM ROUND

COMBINATORICS

1. In how many ways can you create a 4-digit code using the digits 1-9, where each digit is used at most once, and the code must contain at least one even digit?

   Answer: 2904

   Explanation: This question requires complements principle, which means subtracting the ones that doesn't work from total to find the number that works. To find the codes that doesn't work, consider the code that's only made up of odd numbers. Since each digit has to be different, the possibility of odd numbers is 5 x 4 x 3 x 2 = 120. The total number of ways is 9 x 8 x 7 x 6 = 3024, so 3024 - 120 = 2904.

2. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

   Answer: 216

   Explanation: There are total of 4! = 24 ways to place purple balls into grid so that they are in different column and different row. In order to place the green balls, simply consider only one purple ball for each row and column. For the row and column of one purple ball, place green ball so that it goes one for same row and one for same column. That way, the other two green balls are forced. So there are essentially 3 x 3 = 9 ways to place green ball for each purple ball orientation. So in total, 24 x 9 = 216 ways exist.

3. In the Lucky Guy game, a player pays $5 to participate. If the player wins, they receive $30 by correctly guessing the outcome of a coin flip and the location of a ball hidden under one of the three cups. What is the probability that a player will have a total profit of $40 after playing 4 games?

   Answer: 25/216

Explanation:

10 games → 3 wins
(5/6)^2 * (1/6)^2 *($_4C_2$) = 25/216

4. In a card game, there is a deck of 12 unique cards, each labeled with a number from 1 to 12. A player is randomly dealt 4 cards from the deck. The player wins if the sum of the 4 cards is an even number. If the probability of winning is expressed in its simplest fraction form as a/b, what is the value of a + b?

   Answer: 50

   Explanation:

   Choosing 0 odd numbers

   6C4 = 15
   2 odd and 2 even numbers
   6C2 * 6C2 = 225
   4 odd numbers
   6C4 = 15

   total: 12C4 = 495

   255/495 = 17/33

   17+33=50

ALGEBRA

1. It takes Hilly 1 hour and 20 minutes to paint a dorm room, and 50 minutes for her friend to paint the same room. Assuming they work at the same rate, how long will it take for them to paint the room if they work together? Round your answer to the nearest whole number. The unit should be in minutes.

   Answer: 31

   Explanation: Let Hilly's working rate to be H and her friend's working rate to be F. Since it takes Hilly 1 hour and 20 minutes (80 mins) to paint a dorm room, $H = \frac{1}{80}$. In same logic, $F = \frac{1}{50}$. Let t be the time it takes for them to paint the room together. Then, $Ht + Ft = 1$, $\frac{1}{80}t + \frac{1}{50}t = 1$. $\frac{50t+80t}{80(50)} = 1$. Isolating t on one side, we get $t = \frac{4000}{130} \approx 30.769$. Rounding t to the nearest whole number, we get the answer of 31 minutes.

2. Consider the cubic equation $ax^3 + bx^2 + c = 0$ $(a, b, c, \neq 0)$. Each coefficient a, b, c in the equation is a solution to the new equation formed by removing the term containing that coefficient.

For example, $x = c$ is a solution to the equation $ax^3 + bx^2 = 0$.

What is one possible sum of all solutions to the original equation $ax^3 + bx^2 + c = 0$?

(Hint: sum of roots of cubic equation is $-\frac{b}{a}$)

Answer: 1 or -1 (복수정답)

Explanation:

1). For equation $ax^3 + bx^2 = 0$, $x = c$ is the solution → $ac^3 + bc^2 = 0$

2). For equation $ax^3 + c = 0$, $x = b$ is the solution → $ab^3 + c = 0$

` 3). For equation $bx^2 + c = 0$, $x = a$ is the solution → $a^2b + c = 0$

By equating 2). And 3).

$ab^3 =- c$

$a^2b =- c$

$ab^3 = a^2b$

$b^2 = a$

By equating 2). And 3).

$(b^2)b^3 + c = 0$

$b^5 + c = 0$

$c =- b^5$

From 1).

$(b^2)(- b^5)^3 + b(- b^5)^2 = 0$

$b^{11} - b^{17} = 0$

$b^{11}(1 - b^6) = 0$

$b = \pm 1$

When $b = 1 \rightarrow c = -1 \rightarrow a = 1$

When $b = -1 \rightarrow c = 1 \rightarrow a = 1$

Therefore, there are two possible equations:

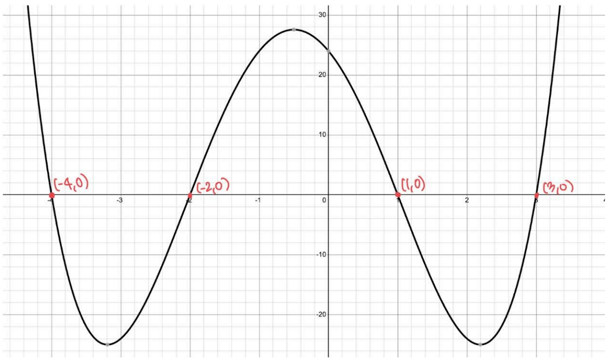
$x^3 + x^2 - 1 = 0$

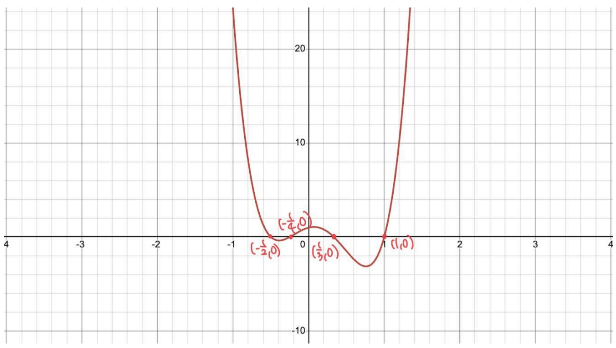
sum of roots = $\frac{-1}{1} = -1$

$x^3 - x^2 + 1 = 0$

sum of roots = $\frac{1}{1} = 1$

3. Given that the equation of the graph on the left is $x^4 + 2x^3 - 13x^2 - 14x + 24$, find the leading coefficient of the equation shown on the right.





Answer: 24

Explanation: From the graph on the left side, we know that $x = -4, -2, 1, 3$ are the roots of the equation $x^4 + 2x^3 - 13x^2 - 14x + 24$. From the graph on the right side, we see that $x = -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, 1$ are the roots of the equation, which are the exact reciprocals of the solutions of the graph shown on the left. Using reciprocal roots formula, we find the equation of the graph on the right side to be $24x^4 - 14x^3 - 13x^2 + 2x + 1$. Thus, the leading coefficient of the equation shown on the right is 24.

NUMBER THEORY

1. Suppose a and b are positive integers larger than 1. Aaron fills up an $a \times b$ table, such that a is the length of horizontal and b is the length of vertical. He fills it up with numbers 1, 2, 3, …, putting the numbers 1, 2, …, b in the first row, $b + 1, b + 2, \ldots$ $2b$ in the second row, and so on, example as shown. Aaron sums up the numbers in his grid after filling it out, and the sum is 595. What is the difference between a and b ?

1	2	3	4
5	6	7	8
9	10	11	12

Answer: 15

Explanation: If a and b are horizontal and vertical, that means ab is the final number of series. The sum would be $\frac{n(n+1)}{2} = \frac{ab(ab+1)}{2} = 595$, so $ab(ab + 1) = 1190$, thus ab and $ab + 1$ have to be the multiple of two consecutive positive integers. After trying out some numbers, 34 and 35 work, meaning $ab = 34$. The only combination of positive integers without either a or b becoming 1 is 2 and 17, which have difference of 15.

2. A positive integer divisor of 15! is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

   Answer: 43

Explanation: The prime factorization of 15! is $2^{11} \cdot 3^{6} \cdot 5^{3} \cdot 7^{2} \cdot 11 \cdot 13$, meaning it has $12 \cdot 7 \cdot 4 \cdot 3 \cdot 2 \cdot 2$ many divisors. Among them, the perfect square integer divisors can be found by putting squared term as one number, like 2^{2} as one designated number. For example, $(2^{2})^{2} \cdot 3^{2}$ is a perfect squared divisor. So, there are total of $6 \cdot 4 \cdot 2 \cdot 2$ many perfect square divisors, and the probability is $\frac{6 \cdot 4 \cdot 2 \cdot 2}{12 \cdot 7 \cdot 4 \cdot 3 \cdot 2 \cdot 2} = \frac{1}{42}$. So $m + n = 43$.

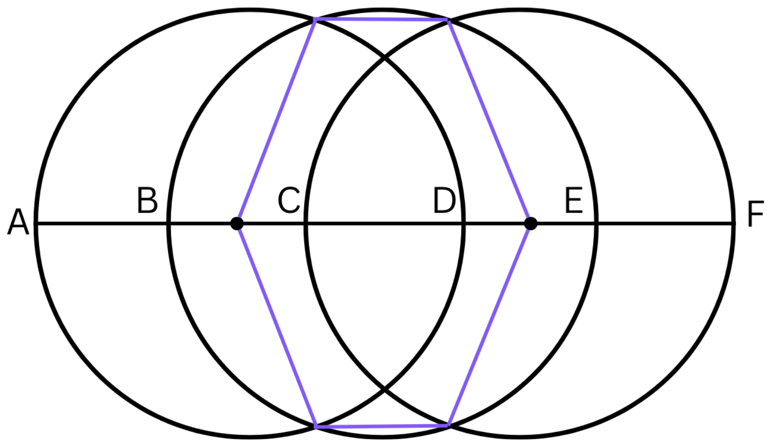
3. A number in base-6 representation is 542. In the base-10 representation, this number is equal to 206. If a number N is equal to 424 in base-6 representation, what is the sum of the number N in base-8 representation and base-10 representation?

   Answer: 400

   Explanation: In the base-10 representation that we use daily, the number N is equal to $4 \cdot 6^{2} + 2 \cdot 6^{1} + 4 \cdot 6^{0} = 160$. To represent this number in base-8, go backwards. $8^{2} = 64$ and there fits two 64s in 160 while not going over, then it's $160 - 64(2) = 32$. There fits 4 8s in 32, so the final base-8 representation of 160 is 240. The sum of 160 and 240 is $160 + 240 = 400$.

GEOMETRY

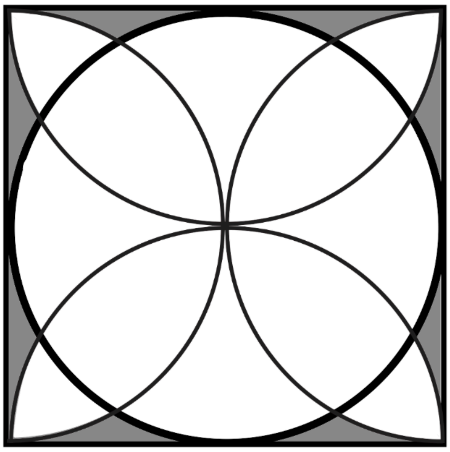
1. Three identical circles with radius of 9 are placed so that they overlap each other and $AB = BC = CD = DE = EF$, as shown. A polygon is drawn by connecting two centers of circle and the intersections of circles. What is the area of the hexagon?



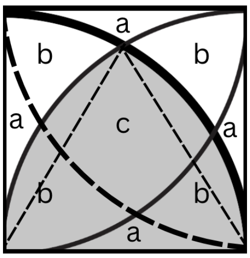
Answer: $108\sqrt{2}$

Explanation: This hexagon is actually 6 congruent triangles put together. One triangle can be drawn by connecting the center of the circle, the intersection of circle 1 and circle 2 and the center of circle 2, which has the area of $18\sqrt{3}$ if you calculate by doing pythagorean theorem and multiplications. So the total is summing up 6 of that, equaling $108\sqrt{2}$.

2. The length of the side of the square is 6. The area of the shaded region can be expressed as $a + b\pi + c\sqrt{3}$. Find $a + b + c$.



Focus on $\frac{1}{4}$ of the figure.



$Equation\ 1: 4a + 4b + c = 3^2$ → Area of the square

$Equation\ 2: 2a + 3b + c = \frac{3^2\pi}{4}$ → Area of the quarter circle

$Equation\ 3: a + 2b + c = 2 * segment -$ A of triangle in dotted line

Equation 2 = 2a + 3b + c = 9π/4 → A of a quarter circle

Equation 3 = a+2b+c = 3π - 9√3/4 → A of orange

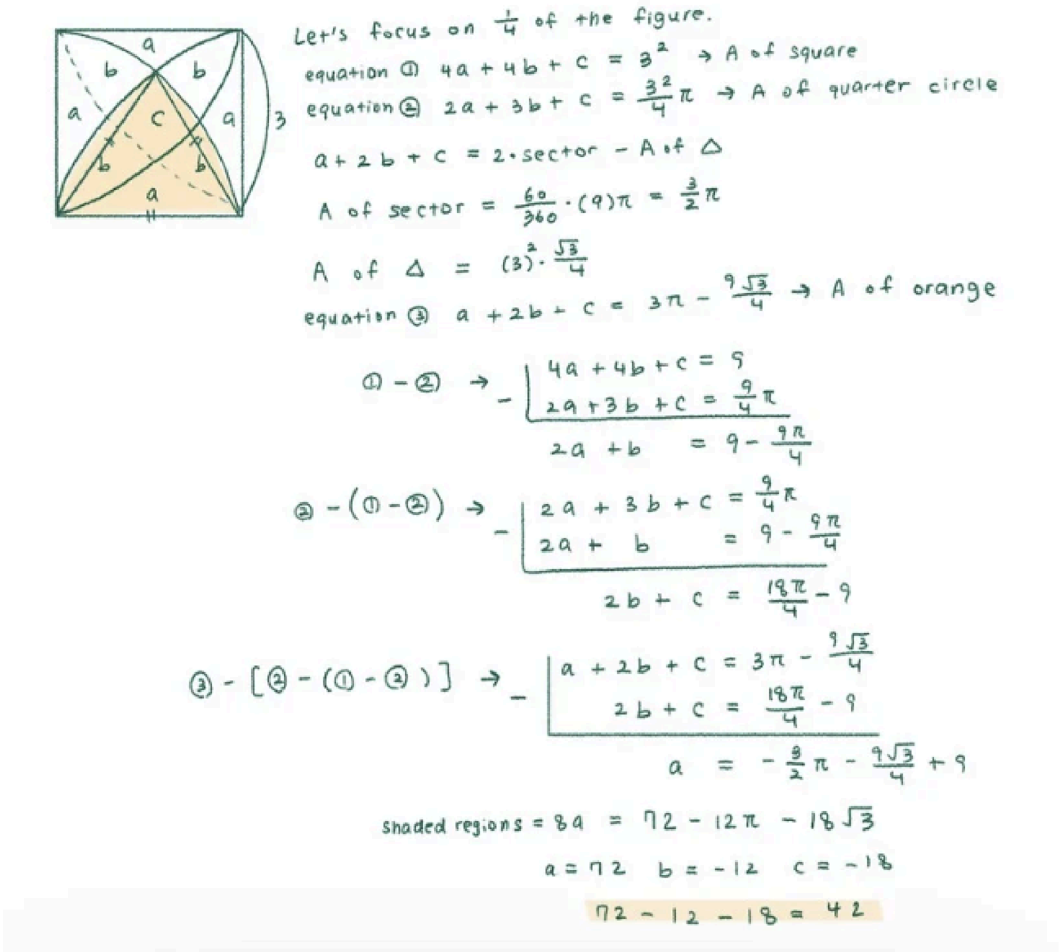
Equation 1 - Equation 2 = 2a + b = 9 - 9π/4

Equation 2 - (Equation 1 - Equation 2) = 2b +c = 18π/4 - 9

Equation 3 -(Equation 2 - (Equation 1 - Equation 2)) = a = -3π/2 - 9√3/4 + 9

Shaded region = 8a = 72 -12π -18√3

72 -12 -18 = 42



COMPLEX NUMBERS / LOGARITHM

1. When $f(z) = z - \bar{z} + 1$ and $f(z + 1) = 6$, solve for $f(\bar{z})$.

Answer: -4

Explanation:

$f(z) = z - \bar{z} + 1$

$f(z + 1) = 6$

$f(z + 1) = z + 1 - \overline{z + 1} + 1$

$= z + 1 - \bar{z} - 1 + 1$

$= z - \bar{z} + 1$

$= 6$

As $z - \bar{z} + 1 = 6, z - \bar{z} = 5$

$f(\bar{z}) = \bar{z} - \bar{\bar{z}} + 1 = \bar{z} - z + 1 = - 5 + 1 = - 4$

2. When positive real number $x, y,$ and z satisfy $log_2{}^x + 2log_4{}^y + 4log_{16}{}^z = 1$, solve for $((2^x)^y)^z$.

   Answer: 4

   Explanation:

   $log_2{}^x + 2log_4{}^y + 4log_{16}{}^z = 1$

   $log_2{}^x + 2log_{2^{\wedge}2}{}^y + 4log_{2^{\wedge}4}{}^z = 1$

   $log_2{}^x + log_2{}^y + log_2{}^z = 1$

   $log_2{}^{xyz} = 1$

   $xyz = 2$

   $((2^x)^y)^z = 2^{xyz} = 2^2 = 4$

3. Find the area surrounded by $y = log_6{}^{(x+1)}$, $y = log_6{}^{(x-1)}$-4, $y = - 2x$, and $y = - 2x + 8$.

   Answer: 16

   Explanation:

   $y = log_6{}^{(x-1)}$-4 is a transformation of $y = log_6{}^{(x+1)}$ to 2 units to the right and 4 units downward. $y = - 2x$ and $y = - 2x + 8$ stays the same even though it transforms 2 units to the right and 4 units downward. So, the area colored is the area highlighted with yellow, which is a parallelogram.

   $x\ intercept\ of\ y = - 2x + 8$: 4 (base)

   $height$: 4

   $Area = base * height = 4 * 4 = 16$