hiSJA MATHEMATICS CONTEST II

April 11, 2025

INTERMEDIATE TEAM ROUND

COMBINATORICS

1. There are 8 Pokpo and 4 Shikmoolji students in a room. Hilly wants to choose 4 people randomly. The probability of choosing an equal number of Pokpo and Shikmoolji students is, in its simplest form, $\frac{a}{b}$. Find $a + b$.

   Answer: 221

   Explanation: There are total of $12C4 = \frac{12(11)(10)(9)}{4(3)(2)(1)}$ ways to randomly choose any 4 people in the room. If we were to choose an equal number of Pokpo and Shikmoolji students, it should be 2 Pokpo students and 2 Shikmoolji students. There are total of $(8C2)(4C2) = \frac{8(7)(4)(3)}{2(2)}$. Since probability $= \frac{\# \text{ of possibilities that satisfy the condition}}{\text{total } \# \text{ of possibilities}}$, the probability of choosing an equal number of Pokpo and Shikmoolji students is $\frac{\frac{8(7)(4)(3)}{2(2)}}{\frac{12(11)(10)(9)}{4(3)(2)(1)}}$. Simplifying, we get $\frac{2(4)(7)}{11(5)(3)} = \frac{56}{165}$. Thus, $a + b = 56 + 165 = 221$.

2. A factor of 320 is chosen randomly, what would be the probability that the factor is even?

   Answer: 6/7

   Explanation:

   2^6 * 5

   12/14 = 6/7

3. In how many ways can the set {a, a, a, b, b, b, b, b, c, c} be arranged?

   Answer: 2520

   10!/3!5!2!

4. Hilly is setting up a new password. His password must include at least one number from 1 to 5, at least one uppercase letter from A to D, and at least one lowercase letter

from a to d. The length of his password should be between 3 to 5 characters. If each letter or number could be used only once, how many valid passwords could he create?

Answer: 451680

Explanation: There are three possible lengths: 3, 4, and 5.

When the length is 3, the number of ways to choose the characters is $5 * 4 * 4 = 80$.

Since the characters could be arranged in any order, the number of possible passwords is $80 * 3! = 80 * 6 = 480$ ways

When the length is 4, the number of ways to choose the characters is $5 * 4 * 4 * 10C1 = 800$.

Since the characters could be arranged in any order, the number of possible passwords is

$800 * 4! = 800 * 24 = 19200$.

When the length is 5, the number of ways to choose the characters is

$5 * 4 * 4 * 10C2 = 3600$

Since the characters could be in any order, the number of possible passwords is

$3600 * 5! = 36800 * 120 = 432000$.

Therefore, the number of valid passwords that could be created is

$480 + 19200 + 432000 = 451680$ ways

5. How many triangles can be formed by connecting vertices in a regular decagon?

   Answer: 120

   Explanation: There are 3 vertices in a triangle. Since a regular decagon has 10 vertices, we have to choose 3 out of the 10.

   $10 C 3 = 120$ triangles.

ALGEBRA

1. The polynomial $x^4 + ax^3 - 7x^2 - x + b$ is divisible by $(x - 2)$ and $(x + 1)$. Find $a + b$.

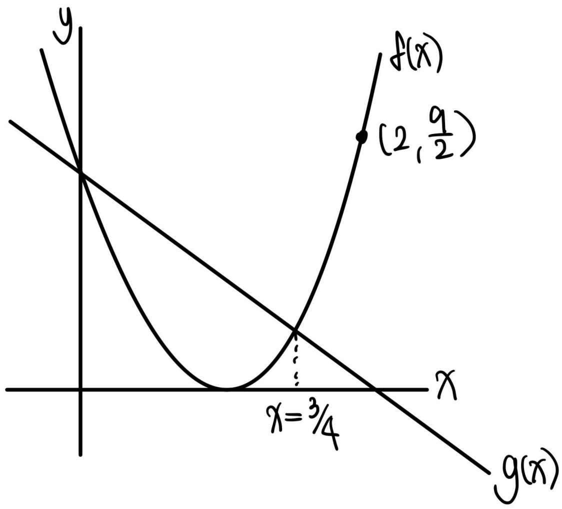
   Answer: 7

   Explanation: Since the polynomial is divisible by 2 and -1, we know that $(2)^4 + a(2)^3 - 7(2)^2 - 2 + b = 0$ and

$(-1)^4 + a(-1)^3 - 7(-1)^2 + 1 + b = 0$. Thus, $a = 1$ and $b = 6$. $a + b = 7$
.

2. Hilly could climb up a ramp at a speed of 2m/s and walk down the ramp at a speed of 5m/s. Her friend could walk up and down the ramp at a speed of 5m/s. The ramp has a base of 30 m and a height of 40 m. They both start at the very bottom of the ramp. When anyone of them reaches the top of the ramp, they turn, and walk down the ramp. If Hilly's friend starts 20 seconds after Hilly starts climbing up the ramp, how long will it take for both to meet each other again?
   Answer: 27.5 sec
   Explanation: Since ramp has a base of 30m and a height of 40m, using the pythagorean theorem, the hypotenuse of the ramp will be 50m. Since Hilly starts 20 seconds faster than her friend, Hilly would have gone 40m up the ramp when her friend starts climbing up the ramp. After 5 seconds, Hilly would have gone up another 10m, reaching the top of the ramp and turning to walk down the ramp at a speed of 5m/s. At that point, her friend would have gone 25 m. Since the distance between them will be 25m, we can set up the equation as $25 = 5t + 5t$, where t represents the time after Hilly begins walking down the ramp. Solving the equation, we get $t = 2.5$. Adding everything in, it will take $20 + 5 + 2.5 = 27.5\, sec$ for both to meet each other again.

3. Two graphs intersect one another as shown in the figure below. Given $f(x) = ax^2 + bx + \frac{1}{2}$ where a and b are both integers, find the leading coefficient of $g(x)$.



Answer: $-\frac{1}{2}$

Explanation: From the graph shown above, we know that $f(x)$ passes through the point $(2, \frac{9}{2})$. Thus, $f(2) = a(2)^2 + b(2) + \frac{1}{2} = \frac{9}{2}$. We also know that the graph of $f(x)$ touches the x-axis, meaning $b^2 - 4a(\frac{1}{2}) = 0$. Using the system of equations, we get $a = \frac{1}{2}$ and $b = 1$, or $a = 2$ and $b = -2$. Since the problem stated that a and b are both integers, $f(x) = 2x^2 - 2x + \frac{1}{2}$. From the figure shown above, we know that $f(x)$ and $g(x)$ interesect at the y-axis. This means that $g(x) = mx + \frac{1}{2}$. We also know that $f(x)$ and $g(x)$ intersect when $x = \frac{3}{4}$. Using $f(x)$ we found before, we can find the y-value: $f(\frac{3}{4}) = 2(\frac{3}{4})^2 - 2(\frac{3}{4}) + \frac{1}{2} = \frac{1}{8}$. This means that $g(\frac{3}{4}) = m(\frac{3}{4}) + \frac{1}{2} = \frac{1}{8}$. Thus, the leading coefficient of $g(x)$, m, is $-\frac{1}{2}$.

4. If $x = 2k$ ($k \neq 0$) is a solution to the quadratic equation $4x^2 + x - 2 = 0$. What is the value of $\frac{k-1}{k^2}$?

   Answer: -8

   Explanation:

   Substitute $x = 2k$ into the given equation

   $4(2k)^2 + 2k - 2 = 0$

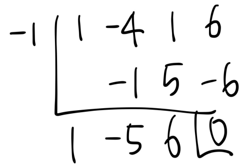
$16k^2 + 2k - 2 = 0$

Divide both sides by $2k^2$

$8 + \frac{1}{k} - \frac{1}{k^2} = 0$

$\frac{1}{k} - \frac{1}{k^2} =- 8$

$\frac{k-1}{k^2} =- 8$

5. Let $f(x) = x^3 - 4x^2 + x + 6$. Find the sum of its greatest root and smallest root.
   Answer: 2
   Explanation: If we try $x = -1$, $f(-1) =- 1 - 4 - 1 + 6 = 0$. So we know that -1 is one of the roots of $f(x)$. Using synthetic division,  , we know that $f(x)$ can be written as $f(x) = (x + 1)(x^2 - 5x + 6)$. Factoring the quadratic section, we get $f(x) = (x + 1)(x - 2)(x - 3)$. Thus, the roots of $f(x)$ will be $x =- 1, 2, 3$. The sum of its greatest root and smallest root is $3 - 1 = 2$.

6. Two trains are on the tracks. Each train is traveling at a constant speed of 75 km/h towards each other. Two trains are initially 300 km apart. A bird starts flying from the front of one train towards the other train at a constant speed of 100 km/h. Once the bird reaches a train, it turns around and flies back towards the first train. This back-and-forth motion continues until the trains collide. Assume that the bird maintains constant speed through its journey. What is the total distance flown by the bird?
   Solution: 200km
   Explanation:
   Two trains are approaching each other at a speed of 75 km/h with an initial gap of 300km.
   This means that two trains meet when the total distance travelled by the both trains is equal to the initial gap, 300km.
   $75t + 75t = 300$

$150t = 300$

$t = 2$ hours

Two trains meet after 2 hours.

Because the bird is flying at a speed of 100km/h, the total distance traveled by the bird is simply its speed times time.

$d = 100 * 2$

$d = 200$ km

NUMBER THEORY

1. The Least Common Multiply of the two numbers is 840, and their Greatest Common Divisor is 14. If one of the numbers is 42, what is the other number?

   LCM * GCD = x * 42
   840 * 14 = 42 * x
   x =280

2. Number a is a factor of 10! and also is a perfect square. What is the sum of the digits of a?

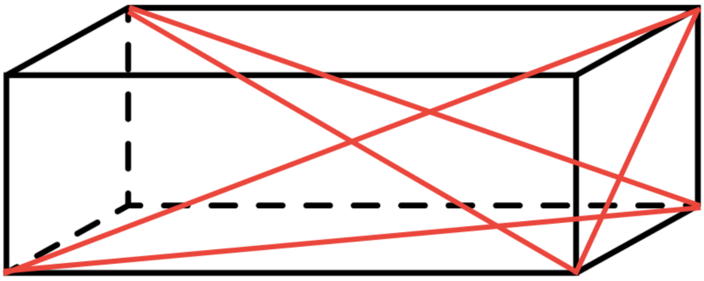
   10! = 1 * 2 * 3 * 2^2 * 5 * 2 * 3 *7 * 2^3 * 3^2 * 2 * 5
   = 2^8 * 3^4 * 5^2 * 7
   → 2^8 * 3^4 * 5^2 = 518400 → 18

3. What is the remainder when 2^{2025} is divided by 5?

   2,4,8,6
   2025/4 =506 … 1
   2

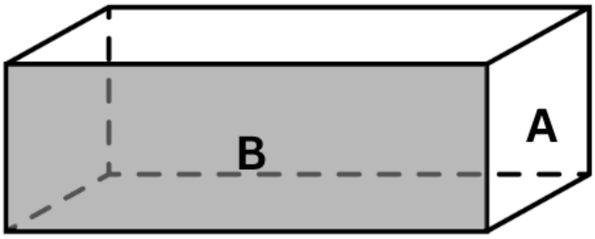
GEOMETRY

1. The given rectangular prism's length, width, and height are 4, 3, and 2 units, respectively. You are making a shape using five pieces of yarn. One piece of yarn is used to connect one vertex to another. What is the average length of the pieces of yarn used to create the following shape?



Answer: $1 + \frac{2\sqrt{29}+2\sqrt{5}+\sqrt{13}}{5}$

Explanation:



Let a represent the longest diagonal, b represent the diagonal of the base, c represent the diagonal of lateral face A, and d represent the diagonal of lateral face B.

The total length of the yarn can be represented by: $2a + b + c + d$.

By the Pythagorean theorem,

$$c = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$d = \sqrt{2^2 + 4^2} = 2\sqrt{5}$$

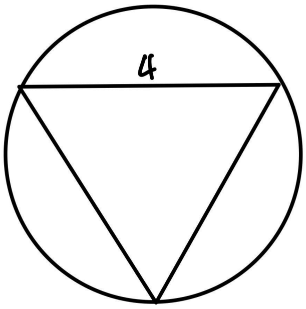
$$b = \sqrt{3^2 + 4^2} = 5$$

$$a = \sqrt{2^2 + 5^2} = \sqrt{29}$$

Since the given shape consists of 5 pieces of yarn, divide the total length of the yarn by 5 to get the average length.

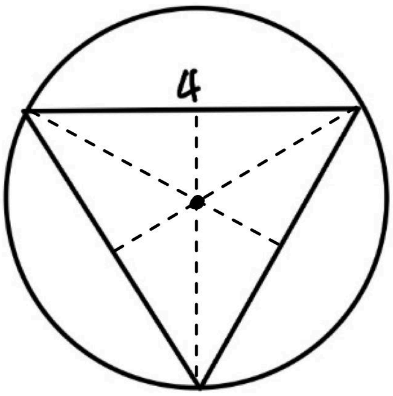
$$\frac{2a+b+c+d}{5} = \frac{2\sqrt{29}+5+\sqrt{13}+2\sqrt{5}}{5} = 1 + \frac{2\sqrt{29}+\sqrt{13}+2\sqrt{5}}{5}$$

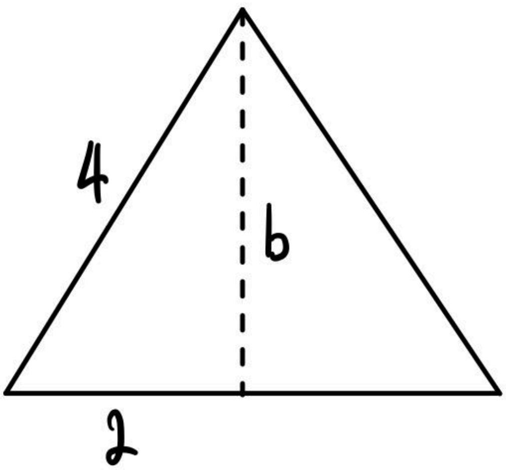
2. The triangle below is an equilateral triangle that is inscribed in a circle. What is the area of the circle?



Answer:$\frac{16\pi}{3}$

Explanation:





By pythagorean theorem:

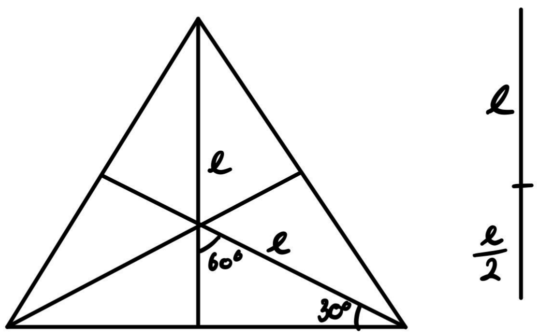
$4^2 = b^2 + 2^2$

$16 = b^2 + 4$

$b = 2\sqrt{3}$

The ratio of altitude is 1:2

Below is a simple proof:



This means that longer portion of the altitude is $\frac{4\sqrt{3}}{3}$ and the shorter portion of the altitude is $\frac{2\sqrt{3}}{3}$.

Because the radius of the circle is equal to the longer portion of the altitude, radius of the circle is $\frac{4\sqrt{3}}{3}$.

Area $= \pi r^2$

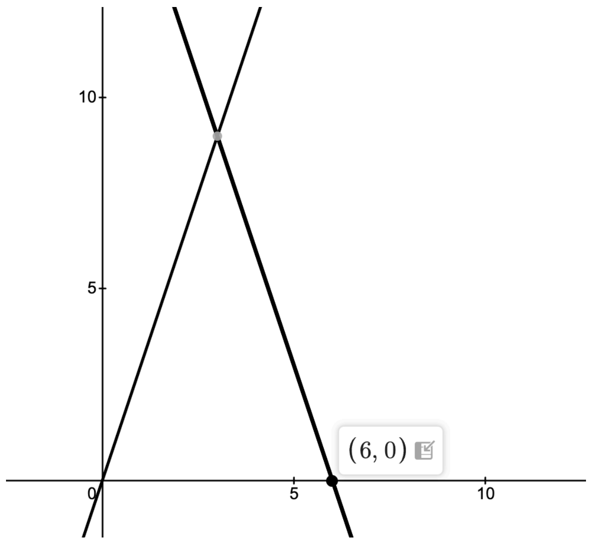
Area $= \pi(\frac{4\sqrt{3}}{3})^2$

Area $= \frac{16\pi}{3}$

3. Line $y = -3x + 18$ forms a right triangle with the positive axes. If line $y = kx$ $(k > 0)$ divides the area of the right triangle in half, find the value of k.

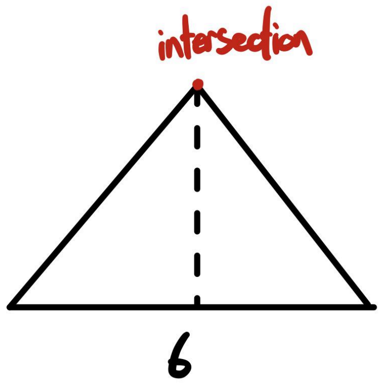
   Answer: 3

   Explanation:



Area $= \frac{1}{2}(6)(18) = 54$

Half of the Area = 27



$kx = -3x + 18$

$(k + 3)x = 18$

$x = \frac{18}{k+3}$

Height = $-3(\frac{18}{k+3}) + 18$

Height = $18 - \frac{54}{k+3}$

Half of the Area (27) has to equal the new smaller triangle

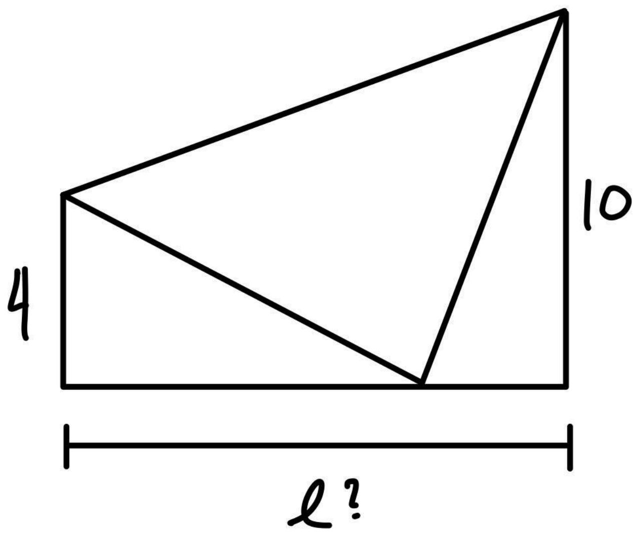
$\frac{1}{2}(6)(18 - \frac{54}{k+3}) = 27$

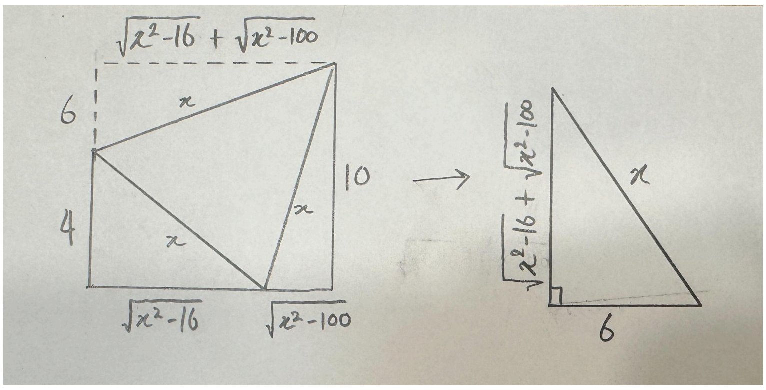
$18 - \frac{54}{k+3} = 9$

$k + 3 = 6$

$k = 3$

4. The triangle below is an equilateral triangle. What is the value of side L?





$$x^2 = 36 + (\sqrt{x^2 - 16} + \sqrt{x^2 - 100})^2$$

$$x^2 = 36 + x^2 - 16 + x^2 - 100 + 2\sqrt{(x^2 - 16)(x^2 - 100)}$$

$$80 - x^2 = 2\sqrt{(x^2 - 16)(x^2 - 100)}$$

$$6400 - 160x^2 + x^4 = 4(x^2 - 16)(x^2 - 100)$$

$$6400 - 160x^2 + x^4 = 4(x^4 - 100x^2 - 16x^2 + 1600)$$

$$6400 - 160x^2 + x^4 = 4x^4 - 464x^2 + 6400)$$

$$3x^4 - 304x^2 = 0$$

$$x^2(3x^2 - 304) = 0$$

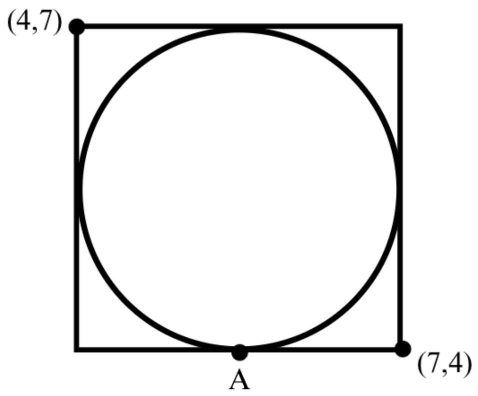
$$x^2 = \frac{304}{3}$$

$$l = \sqrt{x^2 - 16} + \sqrt{x^2 - 100}$$

$$l = \sqrt{\frac{304}{3} - 16} + \sqrt{\frac{304}{3} - 100}$$

$$l = 6\sqrt{3}$$

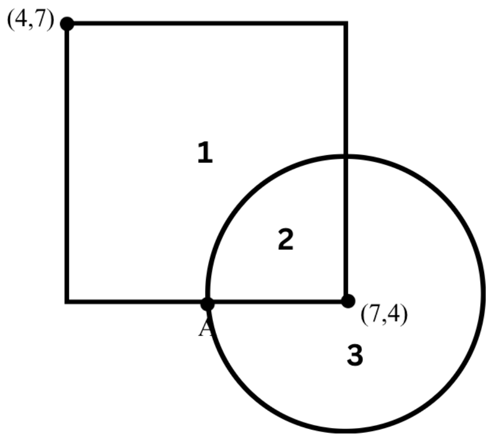
5. A square with a circle inscribed is on a coordinate plane. The circle is rotated $810°$ by point A, resulting in three areas bounded by the shapes. Express the biggest and the smallest area as a ratio.



Answer: $\frac{16}{\pi} - 1$

Explanation:

Since 810°= 2*360°+90°, the figure formed by rotating the circle 810° by point A will be equal to a figure formed by rotating the circle 90° by point A.



After the rotation, the following three regions bounded by the square and the circle are formed.

The coordinates of two of the vertices are given, which makes the side length of the square equal 3. The side length of the square is equal to the diameter of the circle, so the radius of the circle would equal $\frac{3}{2}$.

Region 1 = Area of square - $\frac{1}{4}$(Area of circle)

$= 9 - \frac{1}{4}\left(\frac{9\pi}{4}\right)$

$= 9 - \frac{9\pi}{16}$

Approximate the area of Region 1 by letting $\pi \approx 3$.

Region 1 $\approx \frac{117}{16}$

Region 2 = $\frac{1}{4}$(Area of circle)

$= \frac{9\pi}{16}$

Approximate the area of Region 1 by letting $\pi \approx 3$.

Region 2 $\approx \frac{27}{16}$

Region 3 = $\frac{3}{4}$(Area of circle)

$= \frac{27\pi}{16}$

Approximate the area of Region 1 by letting $\pi \approx 3$.

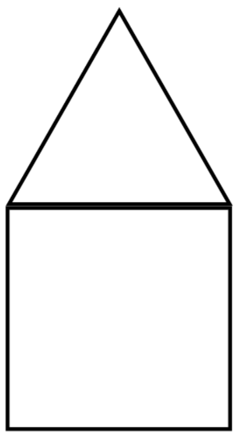
Region 3 $\approx \frac{81}{16}$

∴ Region 1 is the biggest and Region 3 is the smallest.

The ratio of the biggest and the smallest areas can be expressed as:

$$\frac{9-\frac{9\pi}{16}}{\frac{9\pi}{16}} = \frac{144-9\pi}{9\pi} = \frac{16}{\pi} - 1$$

6. The figure is created with an equilateral triangle and a square. The distance between the centroid of the equilateral triangle and the vertex is 9. What is the area of the given figure?



A centroid of a triangle is a point where 3 medians intersect. In an equilateral triangle, the distance from the centroid to the vertex is $\frac{a\sqrt{3}}{3}$, where 'a' represents the sides of the triangle.

$9 = \frac{a\sqrt{3}}{3}$

$a = 9\sqrt{3}$

The side of the triangle equals the side of the square. Find the area of the triangle and the square by calculating $a^2 + \frac{\sqrt{3}}{4}a^2$.

Area = $(9\sqrt{3})^2 + \frac{\sqrt{3}}{4}(9\sqrt{3})^2$

$= 243 + \frac{243\sqrt{3}}{4}$